

A note on corollaries to Tokuyama's Identity for symplectic Schur Q -Functions

Angèle M. Hamel* Ronald C. King†

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Abstract

We present some corollaries to a symplectic primed shifted tableaux version of Tokuyama's identity expressed in terms of other combinatorial constructs, namely generalised U -turn alternating sign matrices and strict symplectic Gelfand-Tsetlin patterns.

1 Introduction

Stimulated by a query from Vineet Gupta, Uma Roy, and Roger Van Peski we present a note on various ways of expressing the type C Tokuyama identity that was first established in its most general form in the context of a symplectic primed shifted tableaux model [4]. This form, Proposition 1.2 of [4], involves parameters $\mathbf{x} = (x_1, x_2, \dots, x_n)$, $\mathbf{y} = (y_1, y_2, \dots, y_n)$ and t . Of these the parameter t carries very little significance, and may be eliminated as shown below in Corollary 2. This result can then be restated in terms of unprimed symplectic shifted tableaux to give Theorem 3. We then show in Section 4 that this theorem may be rewritten in terms of U -turn alternating sign matrices as in Corollary 4 and in terms of strict symplectic Gelfand-Tsetlin patterns as in Corollary 5, where the connection between symplectic shifted tableaux and both U -turn alternating sign matrices and strict symplectic Gelfand-Tsetlin patterns is spelled out in Section 3. Finally the specialisation $y_k = qx_k$ for $k = 1, 2, \dots, n$ is made in section 5, leading to four further type C Tokuyama identities, including one due to Gupta *et al.* [6] given here as our final Corollary 10.

This is intended as a brief note for those already familiar with the ideas and notations. However, for background information and motivation on Tokuyama's identity, see, for example, Tokuyama's original paper [9], Okada's proof and variations [7, 8], and related work of Hamel and King [3, 4], Brubaker, Bump, and Friedberg [1], Gupta, Roy, and Van Peski [5], and Brubaker and Schultz [2], as well as the references therein.

*Department of Physics and Computer Science, Wilfrid Laurier University, Waterloo, Ontario, N2L 3C5, Canada (ahamel@wlu.ca)

†Mathematical Sciences, University of Southampton, Southampton SO17 1BJ, England (R.C.King@soton.ac.uk)

2 Background

For a partition μ of length $\ell(\mu) \leq n$ let $\mathcal{T}^\mu(n)$ be the set of symplectic tableaux T of shape F^μ defined with respect to the alphabet

$$I = \{1 < \bar{1} < 2 < \bar{2} < \dots < n < \bar{n}\}, \quad (1)$$

with the entries of T taken from I :

- T1 weakly increasing across each row from left to right;
- T2 strictly increasing down each column from top to bottom;
- T3 such that k or \bar{k} appear no lower than the k th row.

This is exemplified in the case $n = 4$ and $\mu = (4, 3, 3)$ by

$$T = \begin{array}{|c|c|c|c|} \hline 1 & \bar{1} & 2 & \bar{4} \\ \hline \bar{3} & 4 & 4 & \\ \hline 4 & \bar{4} & \bar{4} & \\ \hline \end{array} \quad (2)$$

The deformed symplectic character $sp_\mu(\mathbf{x}; t)$ is defined by

$$sp_\mu(\mathbf{x}; t) = \sum_{T \in \mathcal{T}^\mu(n)} \text{wgt}(T) \quad (3)$$

where $\text{wgt}(T)$ is the product of weights x_k and $t^2 \bar{x}_k = t^2/x_k$ for each entry k and \bar{k} , respectively, in T .

Then for each strict partition λ of length $\ell(\lambda) = n$ let $\mathcal{ST}^\lambda(n)$ be the set of symplectic shifted tableaux ST of shifted shape SF^λ defined with respect to the same alphabet I with the entries of ST taken from I :

- ST1 weakly increasing across each row from left to right;
- ST2 weakly increasing down each column from top to bottom;
- ST3 strictly increasing down each diagonal from top left to bottom right.

This is exemplified below on the left in the case $n = 5$ and $\lambda = (9, 7, 6, 2, 1)$.

Such tableaux may be refined through the addition of marks or primes. Let $\mathcal{QT}^\lambda(n)$ be the set of primed symplectic shifted tableaux QT obtained from all $ST \in \mathcal{ST}^\lambda(n)$ by adding primes to entries in all possible ways such that

- QT1 any entry identical to its immediate left-hand neighbour must be unprimed;
- QT2 any entry immediately above an identical entry must be primed;
- QT3 any entry with neither of the above neighbours may be either unprimed or primed.

For the given tableau ST shown below on the left, one such corresponding primed tableau QT is shown on the right.

$$ST = \begin{array}{cccccccccc} 1 & \bar{1} & 2 & \bar{2} & 3 & 3 & 4 & \bar{4} & \bar{5} & \\ & 2 & 2 & \bar{2} & \bar{3} & 4 & 4 & \bar{4} & & \\ & & \bar{3} & 4 & \bar{4} & \bar{4} & \bar{4} & \bar{4} & & \\ & & & \bar{4} & \bar{4} & & & & & \\ & & & & 5 & & & & & \end{array} \quad QT = \begin{array}{cccccccccc} 1 & \bar{1} & 2' & \bar{2}' & 3 & 3 & 4' & \bar{4}' & \bar{5} & \\ & 2' & 2 & \bar{2} & \bar{3}' & 4 & 4 & \bar{4}' & & \\ & & \bar{3} & 4 & \bar{4}' & \bar{4} & \bar{4} & \bar{4} & & \\ & & & \bar{4} & \bar{4} & & & & & \\ & & & & 5 & & & & & \end{array} \quad (4)$$

A deformed symplectic Q -function $Q_\lambda(\mathbf{x}/\mathbf{y}; t)$ is then defined by

$$Q_\lambda(\mathbf{x}/\mathbf{y}; t) = \sum_{QT \in \mathcal{QT}^\lambda(n)} \text{wgt}(QT) \quad (5)$$

where $\text{wgt}(QT)$ is the product of the weights $x_k, y_k, t^2 x_k^{-1}$ and $t^2 y_k^{-1}$ for each entry k, k', \bar{k} and \bar{k}' , respectively, in QT .

Our starting point is Proposition 1.2 of [4] which can be written in the form

Proposition 1 *Let μ be a partition of length $\ell(\mu) \leq n$ and $\delta = (n, n-1, \dots, 1)$, so that $\lambda = \mu + \delta$ is a strict partition of length $\ell(\lambda) = n$. Then for all $\mathbf{x} = (x_1, x_2, \dots, x_n)$, $\mathbf{y} = (y_1, y_2, \dots, y_n)$ and t*

$$Q_\lambda(\mathbf{x}/\mathbf{y}; t) = Q_\delta(\mathbf{x}/\mathbf{y}; t) \, sp_\mu(\mathbf{x}; t), \quad (6)$$

with

$$Q_\delta(\mathbf{x}/\mathbf{y}; t) = \prod_{1 \leq i \leq j \leq n} (x_i + y_j)(1 + t^2 x_i^{-1} y_j^{-1}). \quad (7)$$

The parameter t is to some extent redundant. If for all $k = 1, 2, \dots, n$ one maps x_k and y_k to tx_k and ty_k , respectively, so that t^2/x_k and t^2/y_k map to t/x_k and t/y_k , and takes into account the fact that every term in both $\text{wgt}(T)$ and $\text{wgt}(Q)$ of (3) and (5) then carries a factor of t to the power $|\mu|$ and $|\lambda|$, respectively, it follows that in Proposition 1 we have:

$$Q_\lambda(t\mathbf{x}/t\mathbf{y}; t) = t^{|\lambda|} Q_\lambda(\mathbf{x}/\mathbf{y}) \quad Q_\delta(t\mathbf{x}/t\mathbf{y}; t) = t^{|\delta|} Q_\delta(\mathbf{x}/\mathbf{y}); \quad t^{|\mu|} sp_\mu(\mathbf{x}; 1) = sp_\mu(\mathbf{x}), \quad (8)$$

where $Q_\lambda(\mathbf{x}/\mathbf{y}) := Q_\lambda(\mathbf{x}/\mathbf{y}; 1)$, $Q_\delta(\mathbf{x}/\mathbf{y}) := Q_\delta(\mathbf{x}/\mathbf{y}; 1)$ and $sp_\mu(\mathbf{x}) := sp_\mu(\mathbf{x}; 1)$. Then, since $|\lambda| = |\mu| + |\delta|$, we have

Corollary 2 *Let μ be a partition of length $\ell(\mu) \leq n$ and $\delta = (n, n-1, \dots, 1)$, so that $\lambda = \mu + \delta$ is a strict partition of length $\ell(\lambda) = n$. Then for all $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$*

$$Q_\lambda(\mathbf{x}/\mathbf{y}) = \prod_{1 \leq i \leq j \leq n} (x_i + y_j)(1 + x_i^{-1} y_j^{-1}) \, sp_\mu(\mathbf{x}), \quad (9)$$

This $t = 1$ version of Proposition 1 has the advantage of being a Tokuyama-type identity for true, undeformed symplectic characters.

Although both versions have been expressed in terms of primed symplectic shifted tableaux it is a simple matter to write down the equivalent results expressed in terms of unprimed symplectic shifted tableaux. In particular the $t = 1$ version takes the form:

Theorem 3 *Let μ be a partition of length $\ell(\mu) \leq n$ and $\delta = (n, n-1, \dots, 1)$, so that $\lambda = \mu + \delta$ is a strict partition of length $\ell(\lambda) = n$. Then for all $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$*

$$ST_\lambda(\mathbf{x}/\mathbf{y}) = \prod_{1 \leq i \leq j \leq n} (x_i + y_j)(1 + x_i^{-1} y_j^{-1}) \, sp_\mu(\mathbf{x}), \quad (10)$$

where

$$ST_\lambda(\mathbf{x}/\mathbf{y}) = \sum_{ST \in ST^\lambda(\mathbf{x})} \prod_{(i,j) \in SF^\lambda} \text{wgt}(s_{ij}) \quad \text{and} \quad sp_\mu(\mathbf{x}) = \sum_{T \in \mathcal{T}^\mu(\mathbf{x})} \prod_{(i,j) \in F^\mu} \text{wgt}(t_{ij}), \quad (11)$$

with the weights of the entries s_{ij} and t_{ij} at position (i, j) in ST and T , respectively, given by

s_{ij}	$\text{wgt}(s_{ij})$	
k	x_k	if $s_{i,j-1} = k$
	y_k	if $s_{i+1,j} = k$
	$x_k + y_k$	otherwise
\bar{k}	\bar{x}_k	if $s_{i,j-1} = \bar{k}$
	\bar{y}_k	if $s_{i+1,j} = \bar{k}$
	$\bar{x}_k + \bar{y}_k$	otherwise

and

t_{ij}	$\text{wgt}(t_{ij})$
k	x_k
\bar{k}	\bar{x}_k

(12)

Here and elsewhere, for typographical convenience we have set $\bar{x}_k = x_k^{-1}$ and $\bar{y}_k = y_k^{-1}$ for $k = 1, 2, \dots, n$.

3 Symplectic UASMs, GTPs and CPMs

To make contact with other results it is necessary to introduce and relate two combinatorial constructs that are in bijective correspondence with unprimed symplectic shifted tableaux, namely certain U -turn alternating sign matrices (UASMs), and strict symplectic Gelfand-Tsetlin patterns (GTPs).

For each strict partition λ of length $\ell(\lambda) = n$ and breadth $\lambda_1 = m$ let $\mathcal{U}\mathcal{A}^\lambda(n)$ be the set of all $2n \times m$ UASM matrices $A = (a_{ij})$, with $i \in I$ and $j \in J = \{1, 2, \dots, m\}$, whose matrix elements a_{ij} taken from the set $\{1, 0, -1\}$ are such that:

- UA1 the non-zero entries alternate in sign across each row and down each column;
 - UA2 the topmost non-zero entry in any column is 1;
 - UA3 the rightmost non-zero entry in every row is 1;
 - UA4 the sum of entries in each row and in each column is 0 or 1;
 - UA4 $\text{row}_k + \text{row}_{\bar{k}} = 1$ for $k = 1, 2, \dots, n$;
 - UA5 $\text{col}_j = 1$ if $j = \lambda_\ell > 0$ for some ℓ , and $= 0$ otherwise, for $j = 1, 2, \dots, m$,
- where row_i and col_j are the sums of entries in the i th row and j th column of A , respectively, for $i \in I$ and $j \in J$.

An example of such a UASM is illustrated in the case $n = 5$, $m = 9$ and $\lambda = (9, 7, 6, 2, 1)$. Here and elsewhere $\bar{1}$ has been used to denote a matrix element -1 .

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \bar{1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{1} & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{1} & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \bar{1} & 1 & 0 & 0 & 1 & 0 & 0 \\ \bar{1} & 1 & 0 & \bar{1} & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{1} & 1 \end{bmatrix}. \quad (13)$$

A conventional symplectic Gelfand-Tsetlin pattern GT of size $2n \times n$ is an array of non-negative integers m_{ij} of the form

$$GT = \begin{pmatrix} m_{\overline{n}1} & m_{\overline{n}2} & \cdots & \cdots & m_{\overline{n},n} \\ & m_{n1} & m_{n2} & \cdots & m_{n,n-1} & m_{nn} \\ & & \ddots & \ddots & & \vdots \\ & & & m_{\overline{3}1} & m_{\overline{3}2} & m_{\overline{3}3} \\ & & & & m_{31} & m_{32} & m_{33} \\ & & & & & m_{\overline{2}1} & m_{\overline{2}2} \\ & & & & & & m_{21} & m_{22} \\ & & & & & & & m_{\overline{1}1} \\ & & & & & & & & m_{11} \end{pmatrix} \quad (14)$$

subject to the *betweenness conditions*

$$\begin{aligned} m_{\overline{k},j} &\geq m_{k,j} \geq m_{\overline{k},j+1} && \text{for } k = 1, 2, \dots, n-1 \text{ and } j = 1, 2, \dots, k; \\ m_{k+1,j} &\geq m_{\overline{k},j} \geq m_{k+1,j+1} && \text{for } k = 1, 2, \dots, n \text{ and } j = 1, 2, \dots, k, \end{aligned} \quad (15)$$

where $m_{\overline{k},k+1}$ is defined to be 0 for $k = 1, 2, \dots, n$. It follows that each row is a partition.

A symplectic Gelfand-Tsetlin pattern is said to be *strict* if

$$m_{kj} > m_{k,j+1} \quad \text{and} \quad m_{\overline{k}j} > m_{\overline{k},j+1} \quad \text{for } k = 1, 2, \dots, n \text{ and } j = 1, 2, \dots, k-1, \quad (16)$$

in which case each row is a strict partition. For each strict partition λ of length $\ell(\lambda) = n$ let $\mathcal{GT}^\lambda(n)$ be the set of all strict symplectic GTPs GT with top row λ , that is $m_{\overline{n}j} = \lambda_j$ for $j = 1, 2, \dots, n$. We also require that m_{kk} and $m_{\overline{k}k}$ are not both 0 for any $k = 1, 2, \dots, n$. This is to ensure that these GTPs correspond with our ST for which the first entry in the k th row is either k or \overline{k} .

An example of such a strict symplectic GTP is provided in the case $n = 5$ and $\lambda = (9, 7, 6, 2, 1)$ by

$$GT = \begin{pmatrix} 9 & 7 & 6 & 2 & 1 \\ & 8 & 7 & 6 & 2 & 1 \\ & & 8 & 7 & 6 & 2 & 0 \\ & & & 7 & 6 & 2 & 0 \\ & & & & 6 & 4 & 1 & 0 \\ & & & & & 6 & 4 & 3 & 2 \\ & & & & & & 4 & 3 & 2 \\ & & & & & & & 3 & 2 \\ & & & & & & & & 2 & 1 \end{pmatrix} \quad (17)$$

In the case of strict symplectic Gelfand-Tsetlin patterns there are three mutually exclusive possibilities regarding the betweenness conditions. These may either be strict, left or right saturated, but not both. That is to say we can define six sets of propositions:

$$\begin{aligned} B_{kj} &:= m_{kj} > m_{\overline{k-1}j} > m_{k,j+1}; & B_{\overline{k}j} &:= m_{\overline{k}j} > m_{kj} > m_{\overline{k},j+1}; \\ L_{kj} &:= m_{kj} = m_{\overline{k-1}j} > m_{k,j+1}; & L_{\overline{k}j} &:= m_{\overline{k}j} = m_{kj} > m_{\overline{k},j+1}; \\ R_{kj} &:= m_{kj} > m_{\overline{k-1}j} = m_{k,j+1}; & R_{\overline{k}j} &:= m_{\overline{k}j} > m_{kj} = m_{\overline{k},j+1}, \end{aligned} \quad (18)$$

for $1 \leq j < k \leq n$ and

$$\begin{aligned} B_{kk} &:= m_{kk} > 0; & B_{\bar{k}k} &:= m_{\bar{k}k} > m_{kk}; \\ L_{kk} &:= m_{kk} = 0; & L_{\bar{k}k} &:= m_{\bar{k}k} = m_{kk} \\ R_{kk} &:= m_{kk} < 0; & R_{\bar{k}k} &:= m_{\bar{k}k} < m_{kk} \end{aligned} \quad (19)$$

for $1 \leq k \leq n$, such that for all $k = 1, 2, \dots, n$ and $j = 1, 2, \dots, k$

$$\chi(B_{kj}) + \chi(L_{kj}) + \chi(R_{kj}) = 1 \quad \text{and} \quad \chi(B_{\bar{k}j}) + \chi(L_{\bar{k}j}) + \chi(R_{\bar{k}j}) = 1, \quad (20)$$

where χ is the truth function, that is to say $\chi(P) = 1$ if P is true and $= 0$ if P is false for any proposition P . In the case $j = k$ we have

$$\begin{aligned} \chi(R_{kk}) &= \chi(R_{\bar{k}k}) = 0; \\ \chi(B_{kk}) + \chi(L_{kk}) &= 1 \quad \text{and} \quad \chi(B_{\bar{k}k}) + \chi(L_{\bar{k}k}) = 1; \\ \chi(L_{kk}) + \chi(L_{\bar{k}k}) &\leq 1, \end{aligned} \quad (21)$$

where the last condition corresponds to the requirement that m_{kk} and $m_{\bar{k}k}$ are not both 0.

For each strict partition λ of length n the three sets $\mathcal{ST}^\lambda(n)$, $\mathcal{UA}^\lambda(n)$ and $\mathcal{GT}^\lambda(n)$ are in bijective correspondence. Perhaps the simplest bijective correspondence is that between all $GT \in \mathcal{GT}^\lambda(n)$ and all $ST \in \mathcal{ST}^\lambda(n)$. For $GT = (m_{ij})$ this is defined by

$$m_{ij} = \text{the number of entries } \leq i \text{ in row } j \text{ of } ST \quad (22)$$

for $i \in \{1 < \bar{1} < 2 < \bar{2} < \dots < n < \bar{n}\}$ and $j = 1, 2, \dots, n$. Conversely

$$\begin{aligned} \text{the number of entries } k \text{ in row } j \text{ of } ST &= m_{kj} - m_{\overline{k-1}j} \\ \text{the number of entries } \bar{k} \text{ in row } j \text{ of } ST &= m_{\bar{k}j} - m_{kj} \end{aligned} \quad (23)$$

To pass bijectively from $A \in \mathcal{UA}^\lambda(n)$ to $ST \in \mathcal{ST}^\lambda(n)$ one first constructs from A a right-to-left cumulative row sum matrix, $\text{row}(A)$. The non-zero entries 1 in column j of $\text{row}(A)$ then specify by their row number an entry k or \bar{k} on the j th diagonal of ST . To complete the map from $\text{row}(A)$ to ST , these row numbers are arranged in strictly increasing order down each diagonal from top-left to bottom-right.

Similarly, to pass bijectively from $A \in \mathcal{UA}^\lambda(n)$ to $GT \in \mathcal{GT}^\lambda(n)$ one first constructs from A a top-to-bottom cumulative column sum matrix, $\text{col}(A)$. The non-zero entries 1 in row i of $\text{col}(A)$, counted from top to bottom then specify by their column number an entry j in row i of GT counted from bottom to top. These column numbers are arranged in strictly increasing order across each row from left to right starting on the main diagonal with spaces allowing entries in successive rows to lie between one another.

That each of these simple, invertible maps is a bijection is not immediately obvious, but is not hard to verify.

To illustrate the above bijections with our running example, taken from (4), the maps between A , $\text{row}(A)$ and ST are shown below across the top row of (24), and those between A , $\text{col}(A)$ and GT are shown below in the left hand column of (24).

$$\begin{array}{c}
A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \bar{1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{1} & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{1} & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \bar{1} & 1 & 0 & 0 & 0 & 0 & 0 \\ \bar{1} & 1 & 0 & \bar{1} & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{1} & 1 \end{bmatrix} \begin{matrix} 1 \\ \bar{1} \\ 2 \\ \bar{2} \\ 3 \\ \bar{3} \\ 4 \\ \bar{4} \\ 5 \\ \bar{5} \end{matrix} \Leftrightarrow \text{row}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \Leftrightarrow ST = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 1 & \bar{1} & 2 & \bar{2} & 3 & \bar{3} & 4 & \bar{4} & \bar{5} \\ \hline & 2 & \bar{2} & \bar{3} & 4 & \bar{4} & \bar{4} & \bar{4} & \\ \hline & & \bar{3} & 4 & \bar{4} & \bar{4} & \bar{4} & \bar{4} & \\ \hline & & & \bar{4} & \bar{4} & & & & \\ \hline & & & & 5 & & & & \\ \hline \end{array}
\end{array}$$

$$\begin{array}{c}
\Downarrow \quad \swarrow \quad \searrow \\
\text{col}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \quad C(A) = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline \text{WE} & \text{SE} & \text{SE} & \text{SE} & \text{SE} & \text{SE} & \text{SE} & \text{SE} & \text{SE} \\ \hline \text{NS} & \text{WE} & \text{SE} & \text{SE} & \text{SE} & \text{SE} & \text{SE} & \text{SE} & \text{SE} \\ \hline \text{SW} & \text{NW} & \text{WE} & \text{SE} & \text{SE} & \text{SE} & \text{SE} & \text{SE} & \text{SE} \\ \hline \text{SE} & \text{NS} & \text{NW} & \text{WE} & \text{SE} & \text{SE} & \text{SE} & \text{SE} & \text{SE} \\ \hline \text{SE} & \text{SE} & \text{NE} & \text{NS} & \text{SW} & \text{WE} & \text{SE} & \text{SE} & \text{SE} \\ \hline \text{WE} & \text{SE} & \text{NS} & \text{WE} & \text{SE} & \text{NE} & \text{SE} & \text{SE} & \text{SE} \\ \hline \text{NS} & \text{WE} & \text{SE} & \text{NS} & \text{SW} & \text{NW} & \text{WE} & \text{SE} & \text{SE} \\ \hline \text{SW} & \text{NW} & \text{SW} & \text{SW} & \text{SW} & \text{NW} & \text{NW} & \text{WE} & \text{SE} \\ \hline \text{WE} & \text{NE} & \text{SE} & \text{SE} & \text{SE} & \text{NE} & \text{NE} & \text{NE} & \text{SE} \\ \hline \text{NE} & \text{NE} & \text{SE} & \text{SE} & \text{SE} & \text{NE} & \text{NE} & \text{NS} & \text{WE} \\ \hline \end{array} \begin{matrix} 1 \\ \bar{1} \\ 2 \\ \bar{2} \\ 3 \\ \bar{3} \\ 4 \\ \bar{4} \\ 5 \\ \bar{5} \end{matrix}
\end{array}$$

$$\Downarrow$$

$$GT = \begin{pmatrix} 9 & 7 & 6 & 2 & 1 \\ & 8 & 7 & 6 & 2 & 1 \\ & & 8 & 7 & 6 & 2 \\ & & & 7 & 6 & 2 & 0 \\ & & & & 6 & 4 & 1 \\ & & & & & 6 & 3 & 0 \\ & & & & & & 4 & 3 \\ & & & & & & & 3 & 2 \\ & & & & & & & & 2 \\ & & & & & & & & & 2 \\ & & & & & & & & & & 1 \\ & & & & & & & & & & & 1 \end{pmatrix} \begin{matrix} \bar{5} \\ 5 \\ \bar{4} \\ 4 \\ \bar{3} \\ 3 \\ \bar{2} \\ 2 \\ \bar{1} \\ 1 \end{matrix}$$

(24)

In this diagram we have also illustrated a *U*-turn compass point matrix (CPM) $C(A)$ that is obtained from A by mapping the entries 1 and $\bar{1}$ in A to WE and NS, respectively, and the entries 0 in A to one or other of NE, SE, NW or SW in accordance with the arrangements of the nearest non-zero neighbours of the 0 in the four compass point directions, as specified in the following tabulation. In the absence of any non-zero element of A in the directions N or E such a missing element is

deemed to be $\bar{1}$.

UASM entry a_{ij}	1	$\bar{1}$	0	0	0	0
	$\bar{1}$ $\bar{1}$ 1 $\bar{1}$ $\bar{1}$	1 1 $\bar{1}$ 1 1	1 1 $\mathbf{0}$ $\bar{1}$ $\bar{1}$	$\bar{1}$ 1 $\mathbf{0}$ $\bar{1}$ 1	1 $\bar{1}$ $\mathbf{0}$ 1 $\bar{1}$	$\bar{1}$ $\bar{1}$ $\mathbf{0}$ 1 1
CPM entry c_{ij}	WE	NS	NE	SE	NW	SW

(25)

where once again we have adopted a notation whereby $\bar{1}$ signifies an entry -1 , and the notation of [4] has been altered by interchanging both N with S , and W with E .

We define the set $\mathcal{UC}^\lambda(n)$ of CPMs corresponding to $A \in \mathcal{UA}^\lambda(n)$ to be those matrices $C(A)$ obtained in the above manner. They have an important role to play in establishing the equivalence of various possible ways of weighting entries in symplectic shifted tableaux and in strict symplectic Gelfand-Tsetlin patterns.

4 Alternative forms of the symplectic Tokuyama identity

In order to establish corollaries of Theorem 3 within the context of the above combinatorial objects it is merely necessary to replace the sum over $ST \in \mathcal{ST}^\lambda(\mathbf{x}/\mathbf{y})$ by sums over $A \in \mathcal{UA}^\lambda$ and $GT \in \mathcal{GT}^\lambda$, along with appropriate identifications of $\text{wgt}(c_{ij})$ and $\text{wgt}(m_{ij})$.

The simplest case is that of $\mathcal{UA}^\lambda(n)$. The non-zero entries in ST arising from the 1s appearing in $\text{row}(A)$ are associated with the entries WE, SW and NW in $C(A)$. These entries in ST are k or \bar{k} according as the entry in $C(A)$ lies in its row k or \bar{k} , respectively. The fact that the k th entry on the main diagonal of ST is either k or \bar{k} is a consequence of the U -turn nature of A that ensures that either c_{k1} or $c_{\bar{k}1}$ belongs to the set $\{\text{WE}, \text{SW}, \text{NW}\}$, but not both. Moreover the entries SW and NW in $C(A)$ are associated with identical pairs of entries in ST that are horizontal and vertical, respectively. It follows from the tabulation of weights given in (12) for all entries s_{ij} in ST that Theorem 3 may be rewritten in the form:

Corollary 4 *Let μ be a partition of length $\ell(\mu) \leq n$ and $\delta = (n, n-1, \dots, 1)$, so that $\lambda = \mu + \delta$ is a strict partition of length $\ell(\lambda) = n$. Then for all $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$*

$$\sum_{A \in \mathcal{UA}^\lambda(n)} \text{wgt}(A) = \prod_{1 \leq i \leq j \leq n} (x_i + y_j)(1 + x_i^{-1}y_j^{-1}) \text{ sp}_\mu(\mathbf{x}), \quad (26)$$

where

$$\text{wgt}(A) = \prod_{i \in I} \prod_{j=1}^m \text{wgt}(c_{ij}). \quad (27)$$

with

$c_{ij} \ 1 \leq j \leq m$	WE	NS	NE	SE	NW	SW
$i = k$	$x_k + y_k$	1	1	1	y_k	x_k
$i = \bar{k}$	$\bar{x}_k + \bar{y}_k$	1	1	1	\bar{y}_k	\bar{x}_k

(28)

This serves to correct the corresponding result given in Corollary 5.4 of [4]. The correction consists of changing $i < j$ to $i \leq j$ in the first line of equation (5.94), and of replacing $NS_k(A)$ and $NS_{\bar{k}}(A)$ to $WE_k(A)$ and $WE_{\bar{k}}(A)$, respectively, in the third line of equation (5.94). This is necessary to obtain the factorisation as claimed.

It might be noted here that the alternating sign nature of A implies that if $\#XY_i$ is the number of entries XY in row i of $C(A)$ for all pairs of compass point directions X and Y , then

$$\#WE_i = \#NS_i + \chi(P_i) \quad \text{with} \quad P_i := c_{i1} \in \{WE, SW, NW\}. \quad (29)$$

With this notation the U -turn nature of A is reflected in the fact that

$$\chi(P_k) + \chi(P_{\bar{k}}) = 1. \quad (30)$$

Corollary 4 then still applies with the weighting specified by:

$c_{ij} \ 1 \leq j \leq m$	WE	NS	NE	SE	NW	SW
$i = k, j = 1$	$x_k + y_k$	1	1	1	$x_k + y_k$	$x_k + y_k$
$i = \bar{k}, j = 1$	$\bar{x}_k + \bar{y}_k$	1	1	1	$\bar{x}_k + \bar{y}_k$	$\bar{x}_k + \bar{y}_k$
$i = k, j \geq 1$	1	$x_k + y_k$	1	1	y_k	x_k
$i = \bar{k}, j \geq 1$	1	$\bar{x}_k + \bar{y}_k$	1	1	\bar{y}_k	\bar{x}_k

(31)

where associating factors $x_k + y_k$ and $\bar{x}_k + \bar{y}_k$ with all the entries NS rather than WE in $C(A)$ is compensated for by the inclusion of the additional $j = 1$ weights dictated by (29).

Turning to Gelfand-Tsetlin patterns, the analog of Corollary 4 takes the form:

Corollary 5 *Let μ be a partition of length $\ell(\mu) \leq n$ and $\delta = (n, n-1, \dots, 1)$, so that $\lambda = \mu + \delta$ is a strict partition of length $\ell(\lambda) = n$. Then for all $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$*

$$\sum_{GT \in \mathcal{GT}^\lambda(n)} \text{wgt}(GT) = \prod_{1 \leq i \leq j \leq n} (x_i + y_j)(1 + x_i^{-1}y_j^{-1}) \text{ sp}_\mu(\mathbf{x}), \quad (32)$$

where

$$\begin{aligned} \text{wgt}(GT) &= \prod_{k=1}^n \prod_{j=1}^k (\chi(B_{kj})(x_k + y_k) + \chi(L_{kj})x_k + \chi(R_{kj})y_k) x_k^{m_{kj} - m_{\overline{k-1}j} - 1} \\ &\times \prod_{k=1}^n \prod_{j=1}^k (\chi(B_{\bar{k}j})(\bar{x}_k + \bar{y}_k) + \chi(L_{\bar{k}j})\bar{x}_k + \chi(R_{\bar{k}j})\bar{y}_k) \bar{x}_k^{m_{\bar{k}j} - m_{kj} - 1}, \end{aligned} \quad (33)$$

with $m_{\overline{k-1}k} = 0$ for $k = 1, 2, \dots, n$.

Proof: Let ST and A be the elements of $\mathcal{ST}^\lambda(n)$ and $\mathcal{UA}^\lambda(n)$ in bijective correspondence with a given $GT \in \mathcal{GT}^\lambda(n)$, and let $C(A)$ be the corresponding compass point matrix. The relationship (23) between ST and GT implies that the sequence of entries k in row j of ST has length $\ell = m_{kj} - m_{\overline{k-1}j}$. In the case $j < k$ if $\ell > 0$ the leftmost of these entries will be either the beginning of a continuous strip of k s in row j , or the continuation of a continuous strip of k 's from row $j + 1$ with an identical entry k immediately beneath it. All other entries k in row j will have an identical entry k immediately to its left. These two cases correspond to the existence of an entry NS or NW in row k of $C(A)$ together with a sequence of $\ell - 1$ entries SW. However, the map from A to GT by way of $\text{col}(A)$ is such that in the notation of (18) the betweenness conditions $\chi(B_{kj}) = 1$, $\chi(R_{kj}) = 1$ and $\chi(L_{kj}) = 1$ correspond to the existence of an entry NS, NW and NE, respectively, in row k of $C(A)$. It follows from the weighting (31) of $C(A)$ that the weights attached to $\chi(B_{kj})$ and $\chi(R_{kj})$ must be $(x_k + y_k)x_k^{\ell-1}$ and $y_k x_k^{\ell-1}$, respectively, as indicated in the top-line of (33). On the otherhand, if $\ell = 0$ there are no entries k in row j so the contribution to the weight should simply be a factor 1. This is reflected in the fact that in this case $\chi(L_{kj}) = 1$ and the weight attached to it in (33) is indeed just $x_k x_k^{0-1} = 1$, as required.

The case $j = k$ is slightly different. The sequence of entries k in row k of ST now has length $\ell = m_{kk}$, which is of course equal to $m_{kk} - m_{\overline{k-1},k}$ since $m_{\overline{k-1},k}$ has been defined to be 0 by hypothesis. For $\ell > 0$ the leftmost entry k necessarily lies on the main diagonal of ST and carries a weight $x_k + y_k$ with the remaining $\ell - 1$ entries k in row k all carrying a weight x_k . This is correctly codified in (33) since in this case we have $m_{kk} > 0$ so that $\chi(B_{kk}) = 1$ and $\chi(L_{kk}) = \chi(R_{kk}) = 0$. If $\ell = 0$ there is no contribution to $\text{wgt}(GT)$ and this is again in accord with (33) since $\ell = m_{kk} = 0$ implies that $\chi(L_{kk}) = 1$ and $\chi(B_{kk}) = \chi(R_{kk}) = 0$, with the weight attached to $\chi(L_{kk})$ being $x_k x_k^{0-1} = 1$, as required.

Sequences of entries \overline{k} of length $\ell = m_{\overline{k}k} - m_{kk}$ in row j of ST can be dealt with in a similar manner so as to confirm the validity of the second line of (33). If $\ell > 0$ and $j < k$ then the argument is exactly the same as before with k replaced by \overline{k} . If $\ell > 0$ and $j = k$ then the sequence of entries \overline{k} in row k of ST only reach the main diagonal if $m_{kk} = 0$, but, whether or not this is the case, they carry a total weight of $(\overline{x}_k + \overline{y}_k)\overline{x}_k^{\ell-1}$. This is once again in accord with (33) since $\chi(B_{\overline{k}k}) = 1$ and $\chi(L_{\overline{k}k}) = \chi(R_{\overline{k}k}) = 0$. If $\ell = 0$ we have both $\chi(B_{\overline{k}j}) = 0$ and $\chi(R_{\overline{k}j}) = 0$ for all j , leaving $\chi(L_{\overline{k}j}) = 1$. However, this leads as usual to a weight contribution of $\overline{x}_k \overline{x}_k^{0-1} = 1$, again just as required. \square

It might be worth noting that the weight of the various combinatorial entities displayed in (24) are given by

$$\text{wgt}(ST) = \text{wgt}(A) = \text{wgt}(GT) = \frac{(x_1 + y_1)^2(x_2 + y_2)^2(x_3 + y_3)^3(x_4 + y_4)^3(x_5 + y_5)^2}{x_1 x_3 x_4^4 x_5 y_1 y_2 y_3^2 y_4^3 y_5} \quad (34)$$

The tableau from (24) can also be written so the weights associated with each box are clearly visible:

$x_1 + y_1$	$\bar{x}_1 + \bar{y}_1$	y_2	\bar{y}_2	$x_3 + y_3$	x_3	y_4	\bar{y}_4	$\bar{x}_5 + \bar{y}_5$
	$x_2 + y_2$	x_2	$\bar{x}_2 + \bar{y}_2$	$\bar{x}_3 + \bar{y}_3$	$x_4 + y_4$	x_4	\bar{y}_4	
		$\bar{x}_3 + \bar{y}_3$	$x_4 + y_4$	\bar{y}_4	\bar{x}_4	\bar{x}_4	\bar{x}_4	
			$\bar{x}_4 + \bar{y}_4$	\bar{x}_4				
				$x_5 + y_5$				

(35)

5 The specialisation $y_k = qx_k$ for all k

The specialisation $y_k = qx_k$ for $k = 1, 2, \dots, n$ for any non-zero parameter q leads to a considerable simplification of all our identities.

Corollary 6 *Let μ be a partition of length $\ell(\mu) \leq n$ and $\delta = (n, n-1, \dots, 1)$, so that $\lambda = \mu + \delta$ is a strict partition of length $\ell(\lambda) = n$. Then for all $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and q*

$$\sum_{ST \in ST^\lambda(\mathbf{x})} \text{wgt}(ST) = \prod_{1 \leq i \leq j \leq n} (x_i + qx_j)(1 + q^{-1}x_i^{-1}x_j^{-1}) \text{sp}_\mu(\mathbf{x}), \quad (36)$$

where

$$\text{wgt}(ST) = \prod_{(i,j) \in SF^\lambda} \text{wgt}(s_{ij}) \quad (37)$$

with

s_{ij}	$\text{wgt}(s_{ij})$		s_{ij}	$\text{wgt}(s_{ij})$	
k	x_k	if $s_{i,j-1} = k$	\bar{k}	\bar{x}_k	if $s_{i,j-1} = \bar{k}$
	$q x_k$	if $s_{i-1,j} = k$		$(1/q)\bar{x}_k$	if $s_{i-1,j} = \bar{k}$
	$(1+q)x_k$	otherwise		$(1+1/q)\bar{x}_k$	otherwise

(38)

Before proceeding to the next corollary it is convenient to introduce a small lemma

Lemma 7 *Let $C(A)$ be the compass point matrix corresponding to the UASM $A \in \mathcal{A}^\lambda$, and Let $\#XY_i$ be the number of entries XY in the i th row of $C(A)$ for $i = k$ or \bar{k} and $k = 1, 2, \dots, n$. Then*

$$\begin{aligned} \#NS_k + \#NW_k + \#NE_k &= k - 1, \\ \#WE_{\bar{k}} + \#NW_{\bar{k}} + \#NE_{\bar{k}} &= k. \end{aligned} \quad (39)$$

Proof: The first result follows from

$$\#NS_k + \#NW_k + \#NE_k = \sum_{i=1}^{\bar{k}-1} \sum_{j=1}^m a_{ij} = k - 1, \quad (40)$$

where the first step follows from the fact that the tabulation of (25) implies that the column sum in A above the position of each entry XY in row k of C is 1 or 0 according

as XY is or is not in $\{\text{NS}, \text{NW}, \text{NE}\}$. The second step is a consequence of the fact that the row sums row_ℓ and $\text{row}_{\bar{\ell}}$ for consecutive pairs of rows $i = \ell$ and $i = \bar{\ell}$ are such that $\text{row}_\ell + \text{row}_{\bar{\ell}} = 1$ for $\ell = 1, 2, \dots, k-1$.

In the second case we have

$$\#\text{NS}_{\bar{k}} + \#\text{NW}_{\bar{k}} + \#\text{NE}_{\bar{k}} = \sum_{i=1}^k \sum_{j=1}^m a_{ij} = k-1 + \chi(P_k) = k - \chi(P_{\bar{k}}), \quad (41)$$

where the first step is essentially the same as before and the second relies on the fact that the row sum row_k is 1 or 0 according as c_{k1} is or is not $\in \{\text{WE}, \text{SW}, \text{NW}\}$, that is to say according as $\chi(P_k)$, as defined in (29), is 1 or 0. The third step just uses the fact that $\chi(P_k) + \chi(P_{\bar{k}}) = 1$. Finally, if one combines this result with the identity (29) one obtains (39), as required. \square

Now we are in a position to state the following

Corollary 8 *Let μ be a partition of length $\ell(\mu) \leq n$ and $\delta = (n, n-1, \dots, 1)$, so that $\lambda = \mu + \delta$ is a strict partition of length $\ell(\lambda) = n$. Then for all $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and q*

$$\sum_{A \in \mathcal{UA}^\lambda(n)} \text{wgt}(A) = \prod_{1 \leq i \leq j \leq n} (x_i + q x_j) (1 + q^{-1} x_i^{-1} x_j^{-1}) \text{sp}_\mu(\mathbf{x}), \quad (42)$$

where

$$\text{wgt}(A) = c_0 \prod_{i=1}^{2n} \prod_{j=1}^m \text{wgt}(c_{ij}), \quad (43)$$

with $c_0 = 1$ and

$c_{ij} \quad 1 \leq j \leq m$	WE	NS	NE	SE	NW	SW
$i = 2k-1$	$(1+q)x_k$	1	1	1	qx_k	x_k
$i = 2k$	$(1+1/q)\bar{x}_k$	1	1	1	$(1/q)\bar{x}_k$	\bar{x}_k

(44)

or equivalently $c_0 = (1+q)/q^{n(n+1)/2}$ and

$c_{ij} \quad 1 \leq j \leq m$	WE	NS	NE	SE	NW	SW
$i = 2k-1$	x_k	$(1+q)$	1	1	qx_k	x_k
$i = 2k$	\bar{x}_k	$(1+q)$	q	1	\bar{x}_k	\bar{x}_k

(45)

Proof: The first form of the weighting with the overall factor $c_0 = 1$ is a direct consequence of Corollary 4 with y_k set equal to $q x_k$ for all k . The second form arises by writing each factor $(1+1/q)$ in the WE column of the \bar{k} row in the form $(1/q)(1+q)$, and the factor 1 in the NE column of the \bar{k} row in the form $(1/q)q$. This allows one to extract a factor $(1/q)^k$ as a result of Lemma 7. Taking the product over k from 1 to n gives an overall factor of $1/q^{n(n+1)/2}$ and leaves contributions $(1+q)$ and q in the WE and NE columns. Finally, one can use the identity (29) to move the factors $(1+q)$ from the WE to the NS column in rows k and \bar{k} , leading to an overall factor of $(1-q)$ to the power $\sum_{k=1}^n (\chi(P_k) + \chi(P_{\bar{k}})) = n$. \square

Similarly, in the terms of symplectic Gelfand-Tsetlin patterns we have:

Corollary 9 Let μ be a partition of length $\ell(\mu) \leq n$ and $\delta = (n, n-1, \dots, 1)$, so that $\lambda = \mu + \delta$ is a strict partition of length $\ell(\lambda) = n$. Then for all $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and q

$$\sum_{GT \in \mathcal{GT}^\lambda(n)} \text{wgt}(GT) = \prod_{1 \leq i \leq j \leq n} (x_i + q x_j)(1 + q^{-1} x_i^{-1} x_j^{-1}) \text{sp}_\mu(\mathbf{x}), \quad (46)$$

where

$$\text{wgt}(GT) = \prod_{k=1}^n \prod_{j=1}^k (1 + q)^{\chi(B_{kj}) + \chi(B_{\overline{k}j})} q^{\chi(R_{kj}) + \chi(L_{\overline{k}j}) - 1} \mathbf{x}^{\text{xwgt}(GT)} \quad (47)$$

with

$$\mathbf{x}^{\text{xwgt}(GT)} = \prod_{k=1}^n \prod_{j=1}^k x_k^{2m_{kj} - m_{\overline{k}j} - m_{\overline{k-1}j}}, \quad (48)$$

and $m_{\overline{k-1}k} = 0$ for $k = 1, 2, \dots, n$.

Proof: Setting $y_k = q x_k$ for all k in (33) gives

$$\begin{aligned} \text{wgt}(GT) &= \prod_{k=1}^n \prod_{j=1}^k (\chi(B_{kj})(1 + q)x_k + \chi(L_{kj})x_k + \chi(R_{kj})q x_k) x_k^{m_{kj} - m_{\overline{k-1}j} - 1} \\ &\times \prod_{k=1}^n \prod_{j=1}^k (\chi(B_{\overline{k}j})(1 + q)\overline{x}_k + \chi(L_{\overline{k}j})q\overline{x}_k + \chi(R_{\overline{k}j})\overline{x}_k) q^{-1} \overline{x}_k^{m_{\overline{k}j} - m_{kj} - 1} \end{aligned} \quad (49)$$

From this one can see immediately that the \mathbf{x} dependence is given by

$$\prod_{k=1}^n \prod_{j=1}^k x_k^{2m_{kj} - m_{\overline{k}j} - m_{\overline{k-1}j}}. \quad (50)$$

and the q dependence by

$$\prod_{k=1}^n \prod_{j=1}^k (1 + q)^{\chi(B_{kj}) + \chi(B_{\overline{k}j})} q^{\chi(R_{kj}) + \chi(L_{\overline{k}j}) - 1}, \quad (51)$$

as required. \square

This result may be expressed in the alternative form first obtained by Gupta *et al.* [6]:

Corollary 10 Let μ be a partition of length $\ell(\mu) \leq n$ and $\delta = (n, n-1, \dots, 1)$, so that $\lambda = \mu + \delta$ is a strict partition of length $\ell(\lambda) = n$. Then for all $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and q

$$\sum_{GT \in \mathcal{GT}^\lambda(n)} \text{qxwgt}(GT) = \prod_{i=1}^n (q x_i + x_i^{-1}) \prod_{1 \leq i < j \leq n} (x_i + q x_j)(q + x_i^{-1} x_j^{-1}) \text{sp}_\mu(\mathbf{x}), \quad (52)$$

where

$$\text{qxwgt}(GT) = (1+q)^{B(GT)} q^{R_o(GT)+L_e(GT)} \mathbf{x}^{\text{xwgt}(GT)} \quad (53)$$

with

$$\begin{aligned} B(GT) &= \#\{(k,j) \mid m_{k+1,j} > m_{\bar{k}j} > m_{k+1,j+1}, 1 \leq k \leq n-1, 1 \leq j \leq k\} \\ &\quad + \#\{(k,j) \mid m_{\bar{k}j} > m_{kj} > m_{\bar{k},j+1}, 1 \leq k \leq n, 1 \leq j \leq k\}; \\ R_o(GT) &= \#\{(k,j) \mid m_{k+1,j} > m_{\bar{k}j} = m_{k+1,j+1}, 1 \leq k \leq n-1, 1 \leq j \leq k\}; \\ L_e(GT) &= \#\{(k,j) \mid m_{\bar{k}j} = m_{kj} > m_{\bar{k},j+1}, 1 \leq k \leq n, 1 \leq j \leq k-1\}. \end{aligned} \quad (54)$$

with

$$\mathbf{x}^{\text{xwgt}(GT)} = \prod_{k=1}^n \prod_{j=1}^k x_k^{2m_{kj} - m_{\bar{k}j} - m_{\bar{k}-1,j}}, \quad (55)$$

and $m_{\bar{k},k+1} = 0$ for $k = 1, 2, \dots, n$.

Proof: First it should be noted that

$$\begin{aligned} &\prod_{1 \leq i \leq j \leq n} (x_i + q x_j)(1 + q^{-1} x_i^{-1} x_j^{-1}) \\ &= \prod_{i=1}^n (1+q) q^{-1} (q x_i + x_i^{-1}) \prod_{1 \leq i < j \leq n} q^{-1} (x_i + q x_j)(q + x_i^{-1} x_j^{-1}) \\ &= \frac{(1+q)^n}{q^{n(n+1)/2}} \prod_{i=1}^n (q x_i + x_i^{-1}) \prod_{1 \leq i < j \leq n} (x_i + q x_j)(q + x_i^{-1} x_j^{-1}). \end{aligned} \quad (56)$$

Second, that the q -dependence of $\text{wgt}(GT)$ given by (51) can be written in the form

$$\prod_{k=1}^n \left((1+q) \prod_{j=1}^k q^{-1} \right) \times (1+q)^{B(GT)} q^{R_o(GT)+L_e(GT)}, \quad (57)$$

where

$$\begin{aligned} B(GT) &= \sum_{k=1}^n \left(\sum_{j=1}^{k-1} (\chi(B_{kj}) + \chi(B_{\bar{k}j})) + \chi(B_{kk}) \chi(B_{\bar{k}k}) \right); \\ R_o(GT) &= \sum_{k=1}^n \sum_{j=1}^k \chi(R_{kj}); \\ L_e(GT) &= \sum_{k=1}^n \sum_{j=1}^k \chi(L_{\bar{k}j}). \end{aligned} \quad (58)$$

Writing these in the alternative forms displayed in (54), noting that $R_o(GT)$ = the number of NW in odd rows of the CPM, and that $L_e(GT)$ = the number of NE in even rows of the CPM, and cancelling a common factor

$$\prod_{k=1}^n \left((1+q) \prod_{j=1}^k q^{-1} \right) = \frac{(1+q)^n}{q^{n(n+1)/2}} \quad (59)$$

that emerges on both sides of (46) then yields (53). \square

Setting $y_k = qx_k$ for all k in our previous example (34) we find:

$$B(GT) = \#NS = 7, \quad R_o(GT) = \sum_{k=2}^5 \#NW_k = 2, \quad L_e(GT) = \sum_{k=1}^5 \#NE_{\overline{k}} = 5 \quad (60)$$

so that if we exclude the overall factor of $(1+q)^n/q^{n(n+1)/2} = (1+q)^5/q^{15}$ we obtain

$$\text{qxcwt}(GT) = (1+q)^7 q^7 x_2 x_4^{-4}. \quad (61)$$

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References

- [1] B. Brubaker, D. Bump, S. Friedberg, Schur polynomials and the Yang-Baxter equation, *Comm. in Math. Physics* **308** (2011), 1563–1571.
- [2] B. Brubaker, A. Schultz, Deformations of the Weyl character formula for classical groups and the six-vertex model, *J. Algebraic Comb.*, 2015, to appear.
- [3] A.M. Hamel, R. C. King, Symplectic shifted tableaux and deformations of Weyl’s denominator formula for $sp(2n)$, *J. Algebraic Comb.* **16** (2002), 269–300.
- [4] A.M. Hamel, R.C. King, Bijective proofs of shifted tableau and alternating sign matrix identities, *J. Algebraic Comb.* **25** (2007), 417–458.
- [5] V. Gupta, U. Roy and R. Van Peski, A generalization of Tokuyama’s formula to the Hall-Littlewood polynomials, *Electronic J. Combin.* **22** (2015), issue 2, paper #P2.11.
- [6] V. Gupta, U. Roy and R. Van Peski, Conjectural formula for type C , *Private communication*, 2015.
- [7] S. Okada, Partially strict shifted plane partitions, *J. Combin. Theory Ser. A* **53** (1990), 143–156.
- [8] S. Okada, Alternating sign matrices and some deformations of Weyl’s denominator formula, *J. Algebraic Comb.* **2** (1993), 155–176.
- [9] T. Tokuyama, A generating function of strict Gelfand patterns and some formulas on characters of general linear groups, *J. Math. Soc. Japan* **40** (1988), 671–685.